

# Solving the KO Labyrinth

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The *KO Labyrinth*, pictured in FIGURE 1, is a colorful spherical puzzle with 26 chambers, some of which can be connected via internal holes through which a small ball can pass when the chambers are aligned correctly. The puzzle can be realigned by performing physical rotations of the sphere in the same way one manipulates a Rubik's Cube, which alters the configuration of the puzzle. There are two special chambers: one where the ball is put into the puzzle and one where it can exit. The goal is to navigate the ball from the entrance to the exit.



**Figure 1** The *KO Labyrinth*.

We will explore questions related to solving the puzzle, both as originally intended and under modified rules. We first consider a “goal-directed” player who is motivated to reach the end of the maze as quickly as possible and show that the shortest path through the maze takes only 10 moves. Next, we turn to a “random” player who wanders aimlessly through the puzzle. Using two different techniques, we show that such a player makes, on average, about 340 moves before reaching the end of the maze. We

also determine the most- and least-visited chambers. Then, we pose an analogue of the gambler's ruin problem and separately consider whether one can solve the puzzle if we consider certain chambers to be off-limits. We conclude with comments and questions for future investigation.

## The KO graph

The 26 chambers of the puzzle have printed labels consisting of a letter and a number:  $A1$  through  $A8$ ,  $B1$  through  $B12$ , and  $C1$  through  $C6$ . The chambers can be referenced in terms of corners, edges, and faces (as is often done with the Rubik's Cube). The  $A$  chambers are the eight "corners" of the sphere; they can be connected via internal holes only to the  $B$  chambers that are the 12 center "edges." The  $C$  chambers are the six "faces" that can be connected via internal holes only to the  $B$  chambers. There is no central chamber. The ball, via external holes, enters the maze through  $C1$  and exits through  $C6$ .

The *KO Labyrinth* can be manipulated similarly to the Rubik's Cube, which allows for a multitude of puzzle configurations; however, our interest here is not in how the puzzle itself is aligned but rather in how the chambers can be connected. Therefore, we define a *move* to be the act of directly passing the ball through a hole from one chamber to another, where such an act may require a number of Rubik's Cube-like rotations of the actual sphere before it can be executed. Insertion and extraction of the ball through the exterior holes are not considered moves. Not all pairs of chambers can be connected to allow a direct move even when the chambers can be placed next to each other because there is not always a hole. By inspecting the puzzle to determine all possible moves, we generated what we call the *KO graph*, shown in FIGURE 2. The 26 vertices represent the chambers of the puzzle, with two vertices joined by an edge if it is possible to move the ball between those two chambers.

A solution to the puzzle is thus a path from vertex  $C1$  to vertex  $C6$ . More formally, a *path* in a graph is a sequence of distinct vertices where each pair of consecutive vertices is joined by an edge. Recall that the *length* of a path is the number of edges involved and the *distance* between two vertices is the length of a shortest path between them. In addition, a graph is *connected* if every pair of vertices is connected by a path, and *disconnected* otherwise.

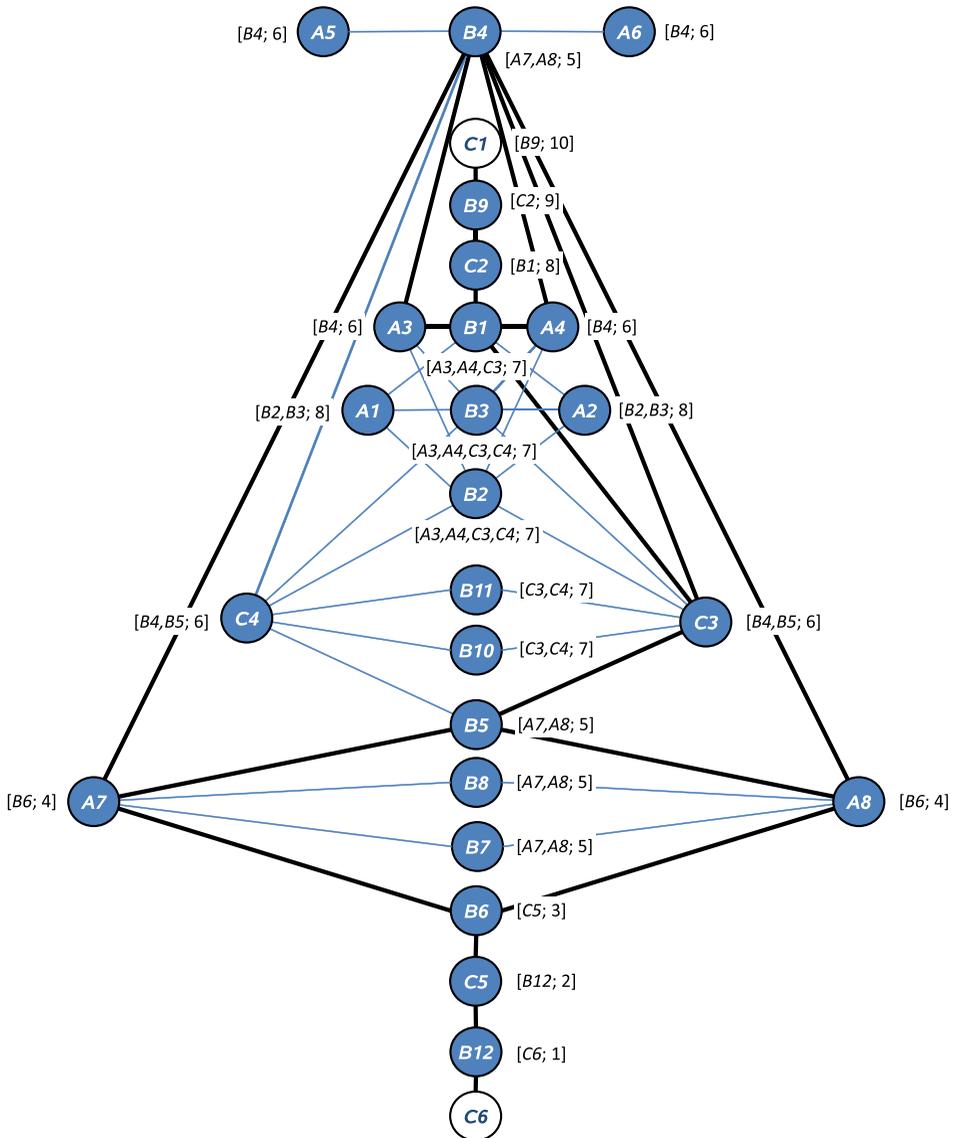
This representation of the KO graph is highly structured, with a vertical axis composed mostly of  $B$  chambers. It is symmetric about this axis except for the edge connecting  $B1$  and  $C3$ . The KO graph is connected.

There are pairs of chambers in the maze that function identically. We call such pairs of chambers "twins." In the KO graph, vertex  $x$  is a *twin* of vertex  $y$  if and only if  $x$  and  $y$  are adjacent to the same vertices. There are seven sets of twin vertices:  $\{A1, A2\}$ ,  $\{A3, A4\}$ ,  $\{A5, A6\}$ ,  $\{A7, A8\}$ ,  $\{B2, B3\}$ ,  $\{B7, B8\}$ , and  $\{B10, B11\}$ .

Now that we have an understanding of the puzzle and a convenient means of representing it mathematically, we will explore its features as a puzzle from two standpoints: goal-directed play and random play.

## The goal-directed player

We begin by considering an intelligent player who has some knowledge of graph theory and uses that to her advantage to navigate through the maze using the least number of moves. We can use a *breadth-first search* or *Dijkstra's algorithm* (see [2], for example) to find the shortest path. Either of these iterative processes can be used to compute the distance between the exit vertex  $C6$  and every other vertex in the KO



**Figure 2** KO graph. The labels are in the form [preceding vertices in the shortest path(s) from C6; distance to C6] and are produced by Dijkstra’s algorithm. The bolded subgraph contains all shortest paths from the entrance chamber C1 to the exit chamber C6.

graph. These distances are listed in Column I of TABLE 1; the distance from C1 to C6 is ten. An advantage to using Dijkstra’s algorithm is that it also enables us, for each vertex, to keep track of the preceding vertex in the shortest path, or vertices if there are multiple shortest paths, from C6. For each vertex in the KO graph in FIGURE 2, the label gives the preceding vertices in the shortest path(s) from C6 and distance to C6. For example, the label next to B1 tells us that it is a distance of seven from C6 and that there are actually at least three shortest paths from B1 to C6 that involve one of A3, A4, or C3.

To determine the total number of ten-move paths, we consider a subgraph of the original KO graph, appearing in bold in FIGURE 2, which is constructed by beginning at C1 and systematically traversing the graph only through the vertices indicated by the

TABLE 1: Results by chamber

- I: Distance to  $C6$  when starting at this chamber  
 II: Expected number of visits to this chamber when starting at  $C1$   
 III: Expected number of moves before reaching  $C6$  when starting at this chamber  
 IV: Probability of reaching  $C6$  in the win-or-lose scenario, starting at this chamber

Chamber	I	II	III	IV
A1	8	12.18	333.33	0.44
A2	8	12.18	333.33	0.44
A3	6	15.97	330.74	0.45
A4	6	15.97	330.74	0.45
A5	6	3.80	322.95	0.47
A6	6	3.80	322.95	0.47
A7	4	17.50	302.00	0.51
A8	4	17.50	302.00	0.51
B1	7	25.24	333.55	0.42
B2	7	23.91	331.72	0.45
B3	7	23.91	331.72	0.45
B4	5	30.39	321.95	0.47
B5	5	14.81	316.05	0.49
B6	3	9.00	261.00	0.58
B7	5	7.00	303.00	0.51
B8	5	7.00	303.00	0.51
B9	9	12.41	341.55	0.14
B10	7	7.81	329.11	0.46
B11	7	7.81	329.11	0.46
B12	1	2.00	89.00	0.86
C1	10	7.21	342.55	–
C2	8	10.41	338.55	0.28
C3	6	27.48	328.60	0.46
C4	6	23.27	327.61	0.46
C5	2	4.00	176.00	0.72

labels produced by Dijkstra's algorithm. We see that there are eight ten-move paths: two by way of  $C3$  and  $B5$  and six by way of  $B4$ .

Two pairs of twins,  $\{A3, A4\}$  and  $\{A7, A8\}$ , appear in these ten-move paths. Twin vertices are redundant in that, in any path, any twin vertex may be replaced by its twin without altering anything else about the path. Removing one twin from  $\{A7, A8\}$  eliminates half of our ten-move paths. Eliminating one twin from both pairs  $\{A3, A4\}$

and  $\{A7, A8\}$  leaves three shortest paths. Even if we eliminate the entire pair  $\{A3, A4\}$ , we still have four ten-move solutions. However, removing the entire pair  $\{A7, A8\}$  causes far more harm as it disconnects the graph and renders the puzzle unsolvable. This observation inspires other questions of connectivity, such as identifying which other chambers are absolutely necessary in order to solve the puzzle, which we will consider later on.

## The random player

We now turn our attention to a simpler player who, not being very insightful, can consider only one move at a time and makes those moves randomly. Recall that we use the word *move* to refer to the KO ball directly passing from one chamber to another and not the number of realignments of the sphere. This simple player has knowledge of only two things: (1) which chamber currently holds the ball (i.e., the current *state*) and (2) what the possible next moves are (i.e., which states the ball can move to). We then wonder: Will the ball ever reach the exit chamber? How many moves, on average, will it take?

**A simulation** There are many ways in which moves can be chosen randomly. We assume our player moves the ball to any adjacent state with equal probability. Each move is made without regard to which moves have occurred previously. This allows for the possibility that the ball may double-back on itself or even travel in a cycle (although the probability that such cycles continue indefinitely is zero). The probabilistic route a player might take along the KO graph constitutes what is known as a *random walk* on the graph, a special case of a Markov chain.

We would like to know how many moves are needed, on average, to complete the maze in a random walk. More generally, we ask for the expected (average) number of moves it takes to get from chamber  $i$  to chamber  $j$  over many independent trials. In answering this, we will use the  $26 \times 26$  *adjacency matrix*  $A$  of our graph, in which the rows and columns correspond to the chambers in the order  $A1$ – $A8$ ,  $B1$ – $B12$ ,  $C1$ – $C6$  and the  $(i, j)$  entry is 1 if vertex  $i$  is adjacent to vertex  $j$ , and 0 otherwise. Note that  $A$  is symmetrical and every diagonal entry is 0.

We can now approximate the answer to our question via the following algorithm:

- (1) Begin with  $i = 21$ , which corresponds to vertex  $C1$ . Set a counter variable  $c = 0$ .
- (2) Randomly generate an integer  $j$  from 1 to 26. This is the vertex that we will try to move to.
- (3) If  $a_{ij} = 0$ , this means that vertex  $i$  and vertex  $j$  are not adjacent, so we cannot move and thus we return to Step 2. If  $a_{ij} = 1$ , we move from vertex  $i$  to vertex  $j$  and increment the counter  $c$  by 1.
- (4) If  $j = 26$ , we have reached the end of the maze (vertex  $C6$ ); we report the value of  $c$  and end the simulation. Otherwise, we replace the value of  $i$  with the value of  $j$  and return to Step 2.

When a simulation ends, the value of the counter  $c$  gives the number of random moves that were required to reach  $C6$  starting in  $C1$ . After one million repeated trials, the average number of moves was 341.79 with a standard deviation of 318.84, and the longest simulation was 3,564 moves. In 162 simulations (0.0162% of the total), one of the ten-move paths was found, a somewhat rare event. In terms of solving the puzzle efficiently, it is clear that random guessing is unlikely to pay off, and clever maneuvers by a goal-directed player are better.

**The end is near** As the number of trials in our simulation increases, our computed average number of moves will approach its theoretical expected value. The theoretical value can be computed directly using some matrix algebra. We form the *transition matrix*  $P$  from the adjacency matrix  $A$  by dividing each entry  $a_{ij}$  by the sum of the entries in the  $i$ th row;  $p_{ij}$  is then the probability that a player currently in state  $i$  will advance to state  $j$  on the next move. Since our player is required to move the ball and not remain in her current chamber, each row of  $P$  sums to 1 and each diagonal element  $p_{ii}$  is 0.

The transition matrix  $P$  represents a nonterminating random walk on a connected graph, which we show the KO graph to be when we consider questions of connectivity later on. As constructed, the KO graph allows a walk that reaches state  $C6$  to leave that state on the next move and continue on indefinitely. However, because reaching state  $C6$  for the first time amounts to solving the puzzle, we modify the KO graph so that the edge leading to state  $C6$  is *unidirectional*, disallowing a return to state  $B12$ . In the language of random walks, this transforms vertex  $C6$  into an *absorbing state* that traps the walk permanently. Any state that is not absorbing is now called a *transient state* and is visited only a finite number of times before the walk is eventually trapped in an absorbing state. The matrix  $P$  is concurrently modified by setting  $p_{26,j} = 0$  for all  $j \neq 26$ , and  $p_{26,26} = 1$ .

How long will it take to reach the absorbing state  $j = 26$ ? This is an example of a *first passage time* problem in which we compute the expected number of moves it takes to “pass into” a state for the first time. We approach the problem by considering how many times we are expected to visit the various transient states before being absorbed. We construct the  $25 \times 25$  submatrix  $Q$  of  $P$  by deleting the 26th row and column of  $P$  so that  $Q$  contains the transient states only. A well-known result [1] from the theory of Markov chains is that the  $(i, j)$  entry of the matrix  $M = (I - Q)^{-1}$  is the expected number of visits to transient state  $j$ , conditional on starting from state  $i$ , before being absorbed. If our player starts at the entrance state  $C1$ , then the corresponding 21st row of  $M$  gives the expected number of visits to each of the remaining states before reaching the end of the maze. These values are given in Column II of TABLE 1.

We note some obvious patterns and symmetries: chamber  $B12$ , the penultimate chamber, is visited the least number of times, and twin chambers such as  $\{A5, A6\}$  are visited the same number of times. The most-visited chamber, with 30.39 visits on average, is  $B4$ , which also happens to be the chamber with the highest degree. And, perhaps frustratingly, we expect to visit chamber  $C1$  (where we began) more than seven times before finally reaching the end!

The row sums of  $M$  provide the expected total number of moves to all transient states before eventually finishing (conditional on starting in the state corresponding to each row), and the sum of the 21st row of  $M$ , 342.55, is the expected number of moves starting from  $C1$ . The mean number of moves computed in our earlier simulation (341.79) can be considered as a single statistical sample (of size  $n = 1,000,000$ ) from a population whose true mean is 342.55. Interestingly, the 95% confidence interval around the sample mean is [341.17, 342.41], which does not capture the true mean, an outcome that should occur in 5% of sampled trials.

What if we start from a state other than the beginning chamber? Intuitively, we expect the total number of moves to be less than 342.55 since we are “closer to the finish line,” and this bears out upon examination of all the row sums of  $M$  (see Column III of TABLE 1). Note the relative constancy of these numbers; starting in almost any chamber, between 302 and 342 random moves are required to reach the end. The exceptions occur if we start in chambers  $B6$ ,  $B12$ , or  $C5$ , which are the closest to the end of the maze. Yet we are perhaps mildly annoyed to see that even if we start in chamber  $B12$  (only one move away!) we are nevertheless expected to take 89 moves to finish. Such is the nature of random decision making.

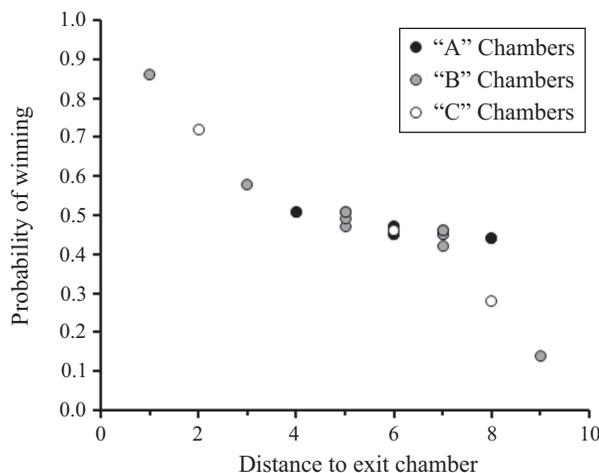
## Changing the rules

Considering variations on a theme arises naturally in many applications and allows for a deeper mathematical exploration. For example, as a way to make the puzzle more challenging, we can consider whether one can still solve the game when certain chambers are off limits. First, however, we explore a novel use of the *KO Labyrinth* by using it to play a simple game modeled after a well-known problem from probability.

**Will I win or will I lose?** In the study of random walks, the classic gambler's ruin problem designates two states as targets to be reached and considers the probability of reaching one before the other. Analogously, we can consider starting the KO ball in the "middle" of the puzzle and play a simple game in which we "win" by reaching the exit chamber *C6* before reaching the entrance chamber *C1* (in which case we "lose"). What, then, is the probability of winning given a particular starting state? To answer this, first designate *C1* as a second absorbing state by removing its corresponding row and column from the submatrix  $Q$  as previously defined to yield  $Q'$ . Next, define the  $24 \times 2$  matrix  $S$  using the rows of  $P$  corresponding to the transient states and the columns of  $P$  corresponding to the absorbing states. Then,  $W = (I - Q')^{-1} S$  is a  $24 \times 2$  matrix in which  $w_{ij}$  is the probability that the ball reaches the  $j$ th absorbing state first, conditional on starting from the  $i$ th transient state.

Column IV of TABLE 1 contains the values  $w_{i2}$ , the probabilities of winning conditional on starting in state  $i$ . The probabilities are roughly the same for a majority of starting states, a consequence of the "forgetfulness" property common to all Markov-type systems, which says that only the current state, and not past history, determines future behavior. Once the player reaches any of the "inner" states in the KO graph, play proceeds as though it had started there. Only initial states near the beginning or end, such as *B9* or *B12*, have a distinct advantage in terms of reaching one particular outcome or another. Interestingly, only 7 out of 24 chambers overall are winning states (wherein one is more likely to reach the end of the puzzle if starting in those chambers) and are, not surprisingly, the ones closest to the exit chamber *C6* in the graph.

How does the probability of winning relate to the distance between a given starting chamber and the exit chamber? In FIGURE 3 we see that the probability of winning



**Figure 3** Probability of winning the gambler's ruin analogue as a function of the starting chamber's distance to the end.

quickly falls to approximately 0.5 as the distance approaches 4. At a distance of 8, we notice an unusual divergence; a player starting in either of the twins  $A1$  and  $A2$  has a much higher probability of winning than one starting in  $C2$ , even though all three are equidistant from  $C6$ . How can this be? We note that  $A1$ ,  $A2$ , and  $C2$  are *not* equidistant from the absorbing (and losing) state  $C1$ . Because we are considering nongol-directed behavior, a random walk starting at  $C2$  is much more likely to be captured by  $C1$  and lose.

**Off limits!** Returning to our goal-directed player, suppose she now wants to make the puzzle more challenging for herself by declaring certain chambers off limits. The natural question to ask is whether this prohibits her from finding a path through the maze. We can address this question in terms of *cut sets*, that is, sets of vertices that, when deleted from the KO graph, result in a disconnected graph.

Connectivity of the KO graph can be determined by visual inspection; however, a more general method especially suitable for larger graphs or for repeated use is to compute the associated *distance matrix*,  $D$ , in which  $d_{ij}$  is the distance from vertex  $i$  to vertex  $j$ :

$$D = A + \sum_{k=2}^{N_v-1} k(R^k - R^{k-1})$$

where  $A$  is our adjacency matrix,  $R = A + I$ ,  $N_v$  is the number of vertices represented in  $A$ , and the matrix powers are computed as *Boolean products*. If  $d_{ij} = 0$  for any  $i \neq j$  then there is no path from  $i$  to  $j$  and the graph is disconnected. For the original KO graph, the entries of the 26th row of  $D$  are the distances produced by Dijkstra's algorithm appearing in Column I of TABLE 1.

Considering cut sets of size  $n = 1$ , a visual inspection of the KO graph reveals that removing any of the singletons  $B9$ ,  $C2$ ,  $B1$ ,  $B6$ ,  $C5$ ,  $B12$ , or  $B4$  disconnects the graph. Moreover, we notice that deleting any of the degree-one vertices  $C1$ ,  $C6$ ,  $A5$ , or  $A6$  does not disconnect the KO graph. Thus, before proceeding to larger cut sets, we make the decision to temporarily remove the eight vertices  $A5$ ,  $A6$ ,  $B9$ ,  $B12$ ,  $C1$ ,  $C2$ ,  $C5$ , and  $C6$  from the KO graph as these can either disconnect, or be disconnected from, the graph trivially and call the subsequent adjacency matrix  $A'$ . Once we have chosen to ignore these vertices, removing the remaining singletons  $B1$ ,  $B4$ , or  $B6$  no longer disconnects the graph.

Moving on to cut sets of size  $n > 1$ , we examine under  $A'$  the  $\binom{18}{n}$  unique sets of  $n$  vertices that can be deleted. Let  $\Delta_n$  be the set of cut sets of size  $n$  and  $|\Delta_n|$  its cardinality. To construct  $\Delta_n$ , we systematically choose each subset of size  $n$  from the graph, modify the adjacency matrix  $A'$  by removing the rows and columns corresponding to that subset, and determine connectedness by computing the associated distance matrix.

Removal of cut sets up to  $n = 5$  produces over 12,000 subgraphs whose connectedness needs to be determined. We can reduce the number of computations required by noticing that any candidate set that is a superset of an already-identified cut set will either not be a cut set itself or be a cut set by virtue of containing a cut set. The latter case is in some ways trivial, and we decide to restrict our attention to "novel" cut sets, requiring that no member of  $\Delta_n$  be a superset of any member of  $\Delta_{n-1}$ . TABLE 2 gives the novel cut sets up to  $n = 5$  (no larger cut sets exist). We note that only one cut set,  $\{C3, C4\}$ , can disconnect the graph but still allow the player to complete the maze. As expected, the vertices having the highest degrees are the ones that repeatedly show up in this table.

TABLE 2: Disconnecting Sets

$n$	$ \Delta_n $	$\Delta_n$ (Cut sets of size $n$ )
2	3	$\{\{A7, A8\}, \{B4, B5\}, \{C3, C4\}\}$
3	1	$\{\{B1, B2, B3\}\}$
4	1	$\{\{B2, B3, B4, C3\}\}$
5	2	$\{\{A1, A2, A3, A4, C3\}, \{A3, A4, B2, B3, C3\}\}$

## Further questions

The KO graph offers undergraduates familiar with graph theory many interesting features for further exploration. For example, notions of the graph's *complexity*, such as its girth, chromatic number, diameter, circumference, variance of vertex degrees, and average distance between vertices can be computed. In addition, while we used the distance matrix to show that the KO graph is connected, students can use an alternate method such as creating a spanning tree of the graph. We can also ask how many total solution paths exist and how many of those do not revisit a vertex. Finally, students can pose questions in the spirit of the Traveling Salesman problem, such as determining the fewest number of vertices that must be removed from the KO graph in order to have a Hamilton path from the entrance chamber  $C1$  to the exit chamber  $C6$ .

Other questions involve the mechanics of the puzzle: In our discussion we have defined a "move" as directly passing the ball from one chamber to another, ignoring the physical rotations required to facilitate such a move. If we now consider them, is it possible to find the least number of rotations necessary to perform each of our moves? If so, can we add these as *weights* on the edges of the KO graph and then use Dijkstra's algorithm to determine the least number of rotations required to solve the puzzle? Finally, those interested in puzzle design might also wonder: Does either the expected number of random moves needed to solve the *KO Labyrinth* or the fact that most chambers are nonwinning states under random play serve as a reasonable measure of the difficulty level of the puzzle? How can we use this analysis to create a more challenging maze, and how does that relate to the complexity properties of the graph?

## REFERENCES

1. G. F. Lawler, *Introduction to Stochastic Processes*. Chapman & Hall, Boca Raton, 2006.
2. A. Tucker, *Applied Combinatorics*. John Wiley & Sons, New Jersey, 2007.

**Summary.** The KO Labyrinth is a colorful spherical puzzle with 26 chambers, some of which can be connected via holes through which a small ball can pass when the chambers are aligned correctly. The puzzle can be realigned by performing physical rotations of the sphere in the same way one manipulates a Rubik's Cube, which alters the configuration of the puzzle. The goal is to navigate the ball from the entrance chamber to the exit chamber. We find the shortest path through the puzzle using Dijkstra's algorithm and explore questions related to connectivity of puzzle with the adjacency matrix, distance matrix, and first passage time analysis. We show that the shortest path through the maze takes only 10 moves, whereas a random walk through the maze requires, on average, about 340 moves before reaching the end. We pose an analogue of the gambler's ruin problem and separately consider whether we are able to solve the puzzle if certain chambers are off limits. We conclude with comments and questions for future investigation.

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